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NATURAL CONVECTION IN A SQUARE CAVITY: A COMPARISON EXERCISE

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SUMMARY

A number of contributed solutions to the problem of laminar natural convection in a square cavity have been compared with what is regarded as a solution of high accuracy. The purposes of this exercise have been to confirm the accuracy of the bench mark solution and to provide a basis for the assessment of the various methods and computer codes used to obtain the contributed solutions.

KEY WORDS Comparison Natural Convection Numerical Methods Validation

INTRODUCTION

During the conference on Numerical Methods in Thermal Problems, which took place in Swansea in July 1979, Jones[1] proposed that buoyancy-driven flow in a square cavity with vertical sides which are differentially heated would be a suitable vehicle for testing and validating computer codes used for a wide variety of practical problems. Such problems include reactor insulation, cooling of radioactive waste containers, ventilation of rooms, fire prevention, solar energy collection, dispersion of waste heat in estuaries and crystal growth in liquids.

Following discussions at Swansea, we invited[2–4] contributions to the solution of the problem described below. A total of 37 contributions* from 30 contributors or groups of contributors in nine countries were received, and we are gratified by the world-wide interest expressed in this project.

This paper summarizes and discusses the main features of the contributions, and provides a quantitative comparison between them and what we believe to be a high accuracy solution suitable for use as a bench mark. Full reprints of all the original contributions have been published separately.[5a] A preliminary version of this paper was presented at a session of the 2nd Conference on Numerical Methods in Thermal Problems.[6] Since that preliminary paper was prepared, several contributors have taken the opportunity to provide improved results and additional information. These further data are given in full in Reference 5b and have also been incorporated into the present paper.

* Not counting multiple contributions, distinguished only by mesh size, from the same contributor(s).

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THE COMPARISON PROBLEM[2]

'Consider the two-dimensional flow of a Boussinesq fluid of Prandtl number 0·71 in an upright square cavity described in non-dimensional terms by $0 \leqslant x \leqslant 1$, $0 \leqslant z \leqslant 1$ with z vertically upwards. Assume that both components of the velocity are zero on all the boundaries, that the boundaries at $z=0$ and 1 are insulated, $\partial T/\partial z=0$, and that $T=1$ at $x=0$ and $T=0$ at $x=1$.

'Calculate the flow and thermal field for Rayleigh numbers, $\beta g \Delta T D^3/\kappa\nu$, of 10^3, 10^4, 10^5 and 10^6.

'The following results should be supplied:

average Nusselt number, $\int_0^1 \frac{\partial T}{\partial x} \mathrm{d}z|_{x=0 \text{ or } 1}$

maximum and minimum local Nusselt numbers on the hot wall, and their location;
maximum vertical velocity on the horizontal mid-plane and its location;
maximum horizontal velocity on the vertical mid-plane and its location;
contour plots of the velocity components and, if available, the stream function, the pressure and the vorticity. These will be used only for a qualitative comparison and will not be used to infer numerical values of these quantities.

'In addition, contributions should include no more than two pages* outlining the method used and giving relevant computational details (grid, computer, c.p.u. time, storage, etc.). To facilitate comparisons the non-dimensionalization used for presentation of results should use the same reference scales as those described in Mallinson and de Vahl Davis.[7]'

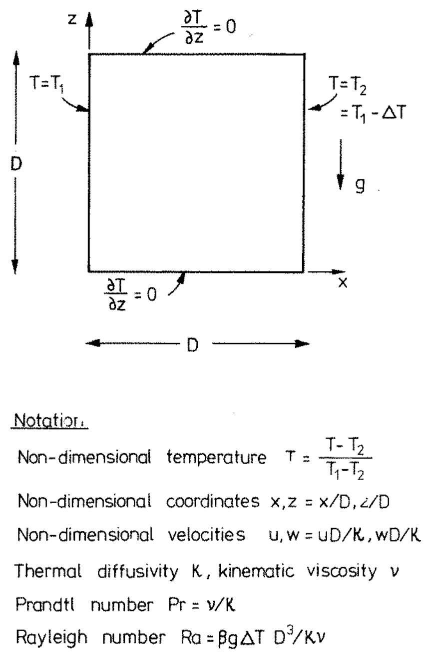

Figure 1. Notation

* A constraint few contributors found themselves able to satisfy!

The notation, as specified in References 3 and 4, is summarized in Figure 1.

After the submissions had been received and given preliminary consideration, we contacted contributors with a request that they supply to us, for each of their solutions, the value of the stream function at the cavity mid-point and the value and location of the maximum stream function (where different from the mid-point value). We again sought information on the c.p.u. time used in obtaining the solutions.

BEST AVAILABLE SOLUTION

One of us (G de VD) used a finite difference (FTCS) method to solve the stream function–vorticity formulation of the equations on successively finer meshes in an attempt to obtain the 'right' answers. Of course, the exact solutions were not found, and the uncertainties increase with Rayleigh number. But the solutions obtained are probably the best currently available, and have been used as a bench mark to assess the contributions received. We are encouraged by the fact that at least two of the contributions agree well with the solutions we are presenting, and we feel that they have the same claim to be called the best.

Our opinion of the quality of the bench mark solution does not imply a judgement of the quality of the particular method employed to achieve it. It reflects the use of mesh refinement and Richardson extrapolation on a convergent method, together with a somewhat generous allocation of computer time.

The method, implemented in a program called FRECON,[8] uses central differences on all spatial derivatives and forward differences on false transient time derivatives in the equations for all three variables (ψ, ζ and T). The finite difference approximations were solved by ADI. The heat flux at the hot (or cold) wall was calculated by a three-point forward (or backward) approximation to $\partial T/\partial x$. The average value $\overline{\mathrm{Nu}}$ was found using Simpson's rule.

Full details of the solution and its derivation are presented in a companion paper.[9] The

Table I. The bench mark solution

	Ra			
	10^3	10^4	10^5	10^6
$\lvert\psi_{mid}\rvert$	1·174	5·071	9·111	16·32
$\lvert\psi\rvert_{max}$	—	—	9·612	16·750
x, z	—	—	0·285, 0·601	0·151, 0·547
u_{max}	3·649	16·178	34·73	64·63
z	0·813	0·823	0·855	0·850
w_{max}	3·697	19·617	68·59	219·36
x	0·178	0·119	0·066	0·0379
$\overline{\mathrm{Nu}}$	1·118	2·243	4·519	8·800
$\mathrm{Nu}_{1/2}$	1·118	2·243	4·519	8·799
Nu_0	1·117	2·238	4·509	8·817
Nu_{max}	1·505	3·528	7·717	17·925
z	0·092	0·143	0·081	0·0378
Nu_{min}	0·692	0·586	0·729	0·989
z	1	1	1	1

bench mark values of the items requested from contributors (but not the contour maps, which are in Reference 9) appear in Table I. It is believed that this solution is in error by no more than 1 per cent at $Ra = 10^6$, and probably by less than a tenth of that at the lower Rayleigh numbers. As discussed below, and also in Reference 9, Nu_{max} at $Ra = 10^6$ may be in error by a slightly greater amount.

Attention is drawn to the fact that the values given in Table I for u_{max}, w_{max}, Nu_{max}, Nu_{min} and ψ_{max} are not necessarily the extreme mesh point values. They were computed by numerical differentiation, using a fourth-order polynomial approximation.

THE CONTRIBUTIONS

Of the 37 contributions received, 21 used second-order finite difference methods (FDM) of which one incorporated a fourth-order deferred correction step; ten used finite element methods (FEM); there were three variations of Hermitian methods in one contribution, one adaptive finite difference method, one Galerkin method, and one use of spline approximations.

Table II summarizes the essential features of the methods used in the various contributions.* The names and addresses of the contributors are given in the Appendix.

All but one of the FEM, and ten of the FDM, solved equations for the primitive variables (PV). Nine FDM and one FEM solved equations for stream function and vorticity ($\psi - \zeta$), as did the Hermitian and spline methods. Two contributions considered the velocity–vorticity equations ($u - \zeta$) and one is based on the biharmonic ψ equation; each of these used FDM.

A variety of mesh sizes and element distributions, both uniform and non-uniform, was used. Several contributors submitted results for two or more mesh (or element) sizes, enabling an indication of the residual truncation errors to be obtained, but none attempted extrapolation to zero mesh/element size. One used an adaptive method based on approximations of different order to estimate the errors. In some contributions, advantage has been taken of symmetry to reduce storage and solution time. The equivalent full mesh has, however, been given in Table II.

As mentioned, contributors were invited to report the amount of c.p.u. time taken to obtain their solutions. It is clear from the response to this request that it is not an easy matter to compare c.p.u. times. Most contributors emphasized that their methods were not optimal and that the times could have been substantially reduced if they had tuned their codes or adjusted their solution parameters. Some supplied the total c.p.u. time to solve for a particular Rayleigh number from a cold start; others used a previous solution at a lower Rayleigh number or on a coarser mesh as an initial estimate, presumably reducing the total c.p.u. time consumed and perhaps also easing difficulties due to instability. Some contributors specified the c.p.u. time per iteration, without giving the total number of iterations required. Others gave the total time for all runs.

Schönauer (private communication) has pointed out that an adaptive method automatically supplies an error estimate. The time for such a method should therefore be compared to the total c.p.u. time spent to obtain results in which the contributor has reasonable confidence, i.e. including runs on coarser grids or with different locations of grid points, etc. Certainly none of the contributors supplied this information.

* The contribution from Cooper and Pepper was, through no fault of either them or us, received after our original closing date and was not included in Reference 6. In order that the numbering system of Reference 6 can be retained, the information about their contribution has been listed at the end of Table II instead of being inserted among the other FDMs.

Other considerations which affect a comparison between c.p.u. times are the compiler used, the size of core available, the amount of I/O, and the computer used (speeds can even vary between computers with the same name and number, depending upon some characteristics of the operating system).

It has not been possible to summarize concisely the information on c.p.u. times which has been supplied to us, nor to draw from it any meaningful conclusions. It is therefore not presented here. Instead, the information may be found in the original contributions and the supplements to some of them.[5a,5b]

Tables III to VI contain the quantitative results requested.* The results have been tabulated to the precision given in the original contributions. In cases where results for two or more mesh or element sizes have been submitted, only those for the finest are reported here.

Because the horizontal boundaries of the cavity are adiabatic it should not, in principle, matter where the overall Nusselt number Nu is calculated. In practice, it does seem to matter,[9] and as contributors presented results at various locations, a comparison between them is affected. The original specification of the problem in References 3 and 4 did not fix where Nu was to be calculated; the choice was left to the contributors. Because of queries received, and an editorial constraint imposed upon us by the *Journal of Fluid Mechanics*, the specification there[2] (which is the one presented here) required that Nu should be calculated on one of the vertical walls (it should not matter which). It is clear, however, that the one-sided formula required for calculating a temperature gradient at a wall is less accurate than the central formula of the same formal order of accuracy which may be used at the mid-plane or on any internal vertical plane. It could be argued that the average of the values calculated at all such vertical planes would be more accurate than that at any one plane; and it is suggested in Reference 9 that the mid-plane value $\mathrm{Nu}_{1/2}$ is better still. For this reason, if some of these various estimates for Nu have been given, that quoted here is either the mean Nusselt number $\overline{\mathrm{Nu}}$ or, if available, the mid-plane value $\mathrm{Nu}_{1/2}$.

As has been noted, the bench mark extreme values of Nu and of velocity were obtained by interpolation. Those given by contributors, on the other hand, were not all computed by interpolation; nor is it always clear from the contributions when interpolation was used. It was found, in generating the bench mark solution, that the interpolated values differed by up to 1 per cent from the closest of the adjacent mesh point values; and in one case—that of w_{max} at $\mathrm{Ra}=10^6$ where the profile is very sharp near the maximum—a discrepancy of 6 per cent was encountered. This injects an uncertainty into the comparisons and emphasizes the potential deficiency of using an extreme mesh point value as a function extreme value.

COMPARISON OF THE CONTRIBUTIONS

It could be regarded as invalid or unfair to compare the different contributions with the bench mark solution for an assessment of their accuracy, since they were computed using a variety of meshes. If a method is convergent, a more accurate solution can always be obtained by mesh refinement. Clearly, it could be argued, a 21×21 solution from contributor X cannot be expected to be as good as a 41×41 solution from contributor Y, particularly if the methods used are more or less the same.

And to a certain extent, that would be true. Accuracy should, perhaps, be compared on

* Not all contributors submitted all the information required.

Table II. Main features of the contributions

Indent. No.	Authors	Method	Mesh (points or elements)	Comments
1	Bertelà	FDM; PV: MAC method; viscous dissipation included?	Uniform 21×21	
2	Cuvelier	FDM; PV; artificial compressibility; fractional time steps	Uniform 11×11, 21×21 and 41×41	No solution at 10^6
3	de Vahl Davis and Leong	FDM; ψ–ζ; 4th order deferred correction; false transient	Uniform; 21×21 at 10^3; also 41×41 at 10^4; also 61×61 at 10^5–10^6	
4	Günther I	FDM; PV; DuFort–Frankel; no convergence test: program terminates after prescribed number of time steps	Uniform (?) 41×41 'similar to that of MAC method'. Some results for coarser meshes, also for 82×82 mesh at Ra = 10^6, but difficulty with lack of symmetry was experienced. Results quoted for 41×41 mesh.	
5	Günther II	FDM; PV; fractional time steps; uses Leith's method for convective terms		
6	Günther III	FDM; PV; conservative MAC-like differencing; ADI		
7	Hackbusch	FDM; biharmonic ψ; multi-grid method	Uniform 17×17 and 33×33	Multi-grid method failed at Ra = 10^6 since unable to find solution with 17×17 mesh
8	Jones, Thompson and Woodhouse I	FDM; ψ–ζ; DuFort–Frankel + cyclic reduction for ψ; first-order ζ boundary condition	Uniform 50×50	
9	Jones, Thompson and Woodhouse II	FDM; ψ–ζ; ADI + cyclic reduction for ψ; first-order ζ boundary condition	Non-uniform 33×33; revised solution	Solution at Ra = 10^6 only
10	Kessler and Oertel I	FDM; PV; DuFort–Frankel + cyclic reduction for p	Uniform 40×40 and 70×30 (?)	Solutions at Ra = 10^3, 10^4 only.

11	Küblbeck and Straub	FDM; ψ–ζ; ADI + cyclic reduction for ψ; second upwind differencing for convection	Uniform; 21 × 21 and 41 × 41 at 10^3, 10^4, also 51 × 51 at 10^5, 10^6	Method claimed to be stable up to Ra = 10^{12}.
12	Le Quere and Humphrey	FDM; PV; modified Imperial College TEACH method; 2nd-order upwind differencing for convection	Non-uniform 21 × 21	Ideal gas assumption, not Boussinesq; properties taken at 273–293 K
13	Linthorst and Schinkel	FDM; PV; modified TEACH: hybrid differencing for convection	Non-uniform 21 × 21	
14	Portier, Fraikin and Arnas	FDM; ψ–ζ; ADI, false transient	Non-uniform 25 × 25	
15	Projahn and Rieger	FDM; ψ–ζ; SIP	Non-uniform (body-fitted non-orthogonal); 21 × 21; also 29 × 29 for Ra $\geqq 10^4$.	
16	Quon	FDM; PV	Non-uniform; 25 × 25 at 10^3; 30 × 30 at 10^4; 40 × 40 at 10^5, 10^6	
17	Rheinländer	FDM; ψ–ζ; ADI; upstream weighted differencing	Uniform 41 × 41	Uses k–ε turbulence model
18	Ruel, Grand and Latrobe	FDM; PV; MAC-like	Non-uniform 21 × 21	Results 'little different' from those using 17 × 17 mesh
19	Schönauer and Raith	FDM; u–ζ; adaptive scheme to find optimum order and step size	Non-uniform; 20 × 20 at 10^3, 10^4; 26 × 21 at 10^5; 35 × 25 at 10^6	Estimates errors <1% at 10^5, ~4% at 10^6
20	Thiele	FDM; u–ζ; Poisson Equations for ζ, u, w and T solved by FFT	Uniform 65 × 65	Solution at Ra = 10^6 is not steady
21	Wong and Raithby	FDM; ψ–ζ; ψ and ζ fields solved simultaneously	Uniform (staggered) 33 × 33	Claims accuracy of 1% at Ra $\leqq 10^5$
22	Betts and Lidder	FEM; PV	Triangular elements, quadratic in u, w, T but linear in p. Uniform 200 elements; non-uniform 200 and uniform 800 at Ra = 10^5.	

Table II. (*contd.*)

Indent. No.	Authors	Method	Mesh (points or elements)	Comments
23	Donea and Giuliani	FEM; PV; explicit fractional step method obtaining a pressure correction by satisfying the continuity condition	Uniform 8×8; non-uniform 8×8 and 16×16 elements; 4-node quadrilateral elements; bilinear in u, w, T; uniform in p	
24	Gartling	FEM; PV; false transient at high Ra	Non-uniform 16×16 isoparametric quadrilateral (8-node) elements	
25	Heinrich, Strada and Zienkiewicz	FEM; PV; momentum and energy equations solved *simultaneously* by Newton–Raphson	Non-uniform; 8×8 at Ra $= 10^3$, 10^4; 10×10 at 10^5, 10^6; 9-node isoparametric quadrilaterals	
26	Laval	FEM; PV; explicit	Uniform 4×4 at Ra $= 10^3$–10^5; non-uniform 8×8 at 10^5–10^6; 9-node isoparametric quadrilaterals; biquadratic in u, w, T; bilinear in p	Essentially same as (23) but higher-order elements.
27	Phuoc and Tanner	FEM; PV	Uniform 5×5 at Ra $= 10^3$; non-uniform 8×8 at 10^4, 10^5; 24-degree-of-freedom quadrilaterals composed of four 18-degree-of-freedom triangles; quadratic in u, w; linear in p, T	Solution for Ra $= 10^6$ was too expensive
28	Stevens	FEM; ψ–ζ; solved as coupled system by Newton–Raphson; no special boundary conditions for the vorticity were needed	Uniform 13×13 and 19×19 nodes at Ra $= 10^3$; non uniform 22×19 nodes at 10^4–10^6 and 28×19 nodes at 10^6; quadrilateral elements divided into two 3rd-order triangles	
29	Upson, Gresho and Lee I	FEM; PV; solved as coupled system by Newton–Raphson	Non-uniform 12×12 elements with grid refinement in corner regions; total of 168 nine-node quadratic elements; biquadratic in u, w, T; bilinear in p	Heat flux by conventional T gradient at element Gauss points

30	Upson, Gresho and Lee II	FEM; PV; solved as coupled system by Newton–Raphson	Non-uniform 12 × 12 elements with grid refinement in corner regions; total of 168 nine-node quadratic elements; biquadratic in u, w, T; bilinear in p	Heat flux by 'consistent flux method' based on global heat balance
31	Winters	FEM; PV; Newton–Raphson	Uniform and non-uniform 13 × 13, 21 × 21, 29 × 29 and 37 × 37 nodes; 6-node triangular elements; biquadratic in u, w, T; bilinear in p	
32	Kessler and Oertel II	Galerkin; PV (but p eliminated). Newton method for modified results	Up to 75 trial functions. Modified later to 180 trial functions, replacing the Beam functions with orthogonal polynomials	
33	Lauriat	Spline ADI; ψ–ζ	Uniform 31 × 31	
34	Roux, Bontoux, Gilly and Grondin I	Hermitian; ψ–ζ; ADI; false transient	Uniform; 21 × 21, 41 × 41 at Ra = 10^3–10^5; also 61 × 61 at 10^3	No solution at Ra = 10^6
35	Roux, Bontoux, Gilly and Grondin II	As for (34) except classical 2nd-order scheme for ζ	Uniform 21 × 21	No solution at Ra = 10^6
36	Roux, Bontoux, Gilly and Grondin III	As for (35) except 2nd-order scheme for T	Uniform 21 × 21	Solution at Ra = 10^3 only
37	Cooper and Pepper	FDM, ψ–ζ, strongly implicit procedure	21 × 21 at 10^3–10^5 51 × 51 at 10^6	

Table III. Summary of contributions at $Ra = 10^3$

No.	First author	Nu_0	Nu_{max}	$@z=$	Nu_{min}	$@z=$	u_{max}	$@z=$	w_{max}	$@x=$	ψ_{mid}	ψ_{max}	$@x=$	$@z=$
1	Bertelà	1·12	1·52	0·025	0·686	0·975	3·65	0·825	3·71	0·175	1·19			
2	Cuvelier	1·112	1·49	0·0	0·69	1·0	3·60	0·8	3·72	0·175				
3	de Vahl Davis	1·116	1·494	0·097	0·697	1·0	3·649	0·813	3·696	0·179	1·174			
4	Günther I	1·126	1·511	0·0	0·6834	1·0	3·645	0·82	3·677	0·18				
5	Günther II	1·118	1·508	0·09	0·6904	1·0	3·648	0·82	3·693	0·18				
6	Günther III	1·120	1·517	0·0	0·6866	1·0	3·645	0·82	3·69	0·18				
7	Hackbusch	1·118	1·506	0·062	0·6921	0·969	3·6490	0·781	3·6900	0·156	1·185			
8	Jones I	1·117	1·5070	0·100	0·692	1·0	3·6507	0·792	3·6971	0·1788	1·179			
9	Jones II													
10	Kessler I	1·12	1·53	0·09	0·69	1·00	3·7	0·80	3·7	0·18				
11	Küblbeck	1·119	1·49	0·0875	0·73	1·0	3·59	0·825	3·65	0·175	1·165			
12	Le Quere	1·112	1·49	0·10	0·70	1·0	3·55	0·81	3·64	0·19				
13	Linthorst	1·120	1·517	0·0822	0·689	1	3·63	0·823	3·68	0·174	1·144			
14	Portier	1·143	1·525	0·089	0·685	1·00	3·66	0·814	3·69	0·178				
15	Projahn	1·14	1·54	0·097	0·678	1·0	3·7	0·825	3·77	0·175				
16	Quon	1·128	1·493	0·0896	0·728	0·9901	3·59	0·8340	3·69	0·1760	1·156			
17	Rheinländer	1·1	1·31	0·10	0·92	1·0	2·1	0·82	2·1	0·19	0·67			
18	Ruel	1·12	1·51	0·069	0·69	0·961	3·3	0·81	3·7	0·19				
19	Schönauer	1·1184	1·5069	0·086	0·6918	1·0	3·6489	0·8131	3·6969	0·1783	1·173			
20	Thiele	1·119	1·511	0·0781	0·6892	1·0	3·649	0·8125	3·691	0·1718				
21	Wong	1·12	1·50	0·109	0·691	1·0	3·63	0·813	3·68	0·180	1·17			
22	Betts	1·12	1·50	0·08(5)	0·69	0·99(5)	3·64	0·81	3·70	0·18	1·174			
23	Donea	1·115	1·5040	0·09375	0·6880	0·96875	3·7288	0·8125	3·757	0·1875	1·18			
24	Gartling	1·118	1·506	0·0797	0·691	0·9892	3·640	0·8240	3·696	0·176	1·174			
25	Heinrich	1·1091	1·4862	0·125	0·6977	1·0	3·4707	0·75	3·4621	0·125				
26	Laval	1·138	1·5352	0·125	0·6908	1·0	3·3804	0·875	3·4855	0·125	1·18			
27	Phuoc	1·13	1·50	0·10	0·68	1·0	3·48	0·8	3·43	0·205				
28	Stevens	1·1155	1·497	0·167			3·66	0·82	3·72	0·17	1·17			
29	Upson I	1·1179	1·5064	0·087	0·6912	0·998	3·656	0·812	3·7040	0·166	1·1754			
30	Upson II	1·1178	1·5065	0·0746	0·6912	1·0	3·656	0·812	3·7040	0·166				
31	Winters	1·12	1·51	0·092	0·691	1	3·64	0·81	3·69	0·18	1·175			
32	Kessler II	1·118	1·507	0·086	0·691	1·00	3·649	0·812	3·697	0·178	1·175			
33	Lauriat	1·118	1·515	0·086	0·687	0·992	3·664	0·813	3·7114	0·179				
34	Roux I	1·1181	1·509	0·0833	0·6896	1	3·644	0·8167	3·690	0·1833	1·173			
35	Roux II	1·1154	1·496	0·0	0·6967	1	3·627	0·800	3·653	0·200	1·171			
36	Roux III	1·1249	1·545	0·1	0·6827	1	3·622	0·800	3·648	0·200	1·169			
37	Cooper	1·12	1·52	0·0	0·680	1·0	3·75	0·812	3·63	0·175	1·17			

Table IV. Summary of contributions at $Ra = 10^4$

No.	First author	Nu_0	Nu_{max}	$@z=$	Nu_{min}	$@\ z=$	u_{max}	$@\ z=$	w_{max}	$@\ x=$	ψ_{mid}	ψ_{max}	$@\ x=$	$@\ x=$
1	Bertelà	2·30	3·72	0·125	0·58	0·975	16·14	0·825	19·58	0·125	5·12			
2	Cuvelier	2·268	3·53	0·150	0·54	1.	15·65	0·825	19·48	0·125				
3	de Vahl Davis	2·243	3·507	0·147	0·589	1	16·20	0·823	19·64	0·119	5·081			
4	Günther I	2·248	3·524	0·16	0·5748	1	16·14	0·82	19·55	0·11				
5	Günther II	2·249	3·556	0·13	0·5820	1	16·11	0·82	19·52	0·11				
6	Günther III	2·256	3·561	0·13	0·5914	1	16·10	0·82	19·52	0·11				
7	Hackbusch	2·251	3·568	0·125	0·5839	0·969	16·17	0·781	19·54	0·094	5·128			
8	Jones I	2·242	3·55	0·146	0·586	1·0	16·232	0·823	19·63	0·120	5·110			
9	Jones II													
10	Kessler I	2·27	3·67	0·13	0·63	1·00	16·1	0·83	19·5	0·12				
11	Küblbeck	2·292	3·65	0·125	0·65	1	16·24	0·825	19·32	0·125	5·14			
12	Le Quere	2·232	3·55	0·15	0·61	1	15·55	0·82	19·17	0·13				
13	Linthorst	2·257	3·570	0·126	0·589	1	16·03	0·823	19·50	0·111	4·948			
14	Portier	2·368	3·657	0·141	0·578	1·00	16·19	0·823	19·53	0·119				
15	Projahn	2·27	3·64	0·15	0·59	1·0	16·2	0·883	19·56	0·11				
16	Quon	2·265	3·437	0·1729	0·680	0·9940	16·10	0·8271	19·55	0·1063	5·053			
17	Rheinländer	2·2	4·63	0·10	0	1·0	21·0	0·82	26·3	0·12	7·0			
18	Ruel	2·26	4·41	0·104	0·72	0·988	16	0·81	20	0·1				
19	Schönauer	2·2484	3·5408	0·144	0·5842	1·0	16·19	0·8231	19·545	0·1187	5·072			
20	Thiele	2·260	3·596	0·1406	0·5749	1·0	16·29	0·8281	19·43	0·1250				
21	Wong	2·24	3·53	0·141	0·589	1·0	16·1	0·825	19·5	0·119	5·06			
22	Betts	2·29	3·60	0·15	0·59	1·00	16·10	0·82	19·91	0·12	5·072			
23	Donea	2·2682	3·5392	0·125	0·5824	1	16·698	0·8125	20·032	0·125	5·20			
24	Gartling	2·250	3·538	0·1325	0·587	0·9892	16·186	0·8240	19·630	0·1190	5·074			
25	Heinrich	2·1949	3·4205	0·125	0·5839	1·0	15·8553	0·875	19·9349	0·125				
26	Laval	2·4736	3·7476	0·125	0·6432	1·0	15·238	0·875	19·961	0·125	5·105			
27	Phuoc	2·22	3·51	0·15	0·46	1·0	16·4	0·825	19·5	0·145				
28	Stevens	2·241	3·53	0·145			16·2	0·80	19·7	0·12	5·09			
29	Upson I	2·2479	3·5351	0·15	0·5852	0·998	16·193	0·822	19·675	0·1187	5·0753			
30	Upson II	2·2449	3·535	0·1315	0·5850	1·0	16·193	0·822	19·675	0·1187				
31	Winters	2·25	3·54	0·16	0·585	1	16·2	0·82	19·7	0·12	5·075			
32	Kessler II	2·245	3·527	0·145	0·586	1·0	16·187	0·822	19·628	0·119	5·074			
33	Lauriat	2·247	3·580	0·138	0·578	0·991	16·266	0·8227	19·917	0·1182				
34	Roux I	2·2502	3·546	0·1333	0·5823	1	16·16	0·8167	19·62	0·1167	5·071			
35	Roux II	2·2454	3·546	0·150	0·5892	1	15·85	0·800	19·24	0·100	5·034			
36	Roux III													
37	Cooper	2·26	3·63	0·15	0·563	1·0	16·0	0·83	19·1	0·125	5·09			

Table V. Summary of contributions at $Ra = 10^5$

No.	First author	Nu_0	Nu_{max}	@ $z =$	Nu_{min}	@ $z =$	u_{max}	@ $z =$	w_{max}	@ $x =$	ψ_{mid}	ψ_{max}	@ $x =$	@ $z =$
1	Bertelà	4·90	9·02	0·075	0·69	0·975	34·65	0·875	66·61	0·075	9·40	9·89		
2	Cuvelier	4·799	8·69	0·075	0·39	1·0	33·52	0·9	65·80	0·05				
3	de Vahl Davis	4·517	7·605	0·083	0·739	1·0	34·81	0·855	68.68	0·066	9·121	9·624	0·285	0·602
4	Günther I	4·571	7·718	0·09	0·6791	1	39·44	0·87	70·11	0·06				
5	Günther II	4·565	7·942	0·09	0·7116	1	34·69	0·87	67·78	0·06				
6	Günther III	4·537	7·852	0·06	0·7429	1	35·07	0·87	67·87	0·06				
7	Hackbusch	4·602	8·194	0·062	0·7135	0·969	35·17	0·812	68·00	0·031	9·401	10·0		
8	Jones I	4·523	7·816	0·088	0·734	1·0	35·36	0·855	68·648	0·067	9·281	9·80	0·28	0·6
9	Jones II													
10	Kessler I													
11	Küblbeck	4·638	8·37	0·0625	0·93	1	36·74	0·86	67·90	0·06	9·698	10·101	0·300	0·600
12	Le Quere	4·470	7·82	0·09	0·82	1	36·35	0·85	68·44	0·07				
13	Linthorst	4·535	7·713	0·082	0·790	1	34·83	0·874	68·88	0·065	9·155	9·522	0·263	0·608
14	Portier	4·869	8·386	0·075	0·705	1·00	35·16	0·857	68·14	0·067				
15	Projahn	4·66	8·09	0·067	0·73	1·0	34·8	0·873	68·8	0·061				
16	Quon	4·538	7·569	0·0865	0·808	0·9956	34·81	0·8455	68·26	0·0683	9·104	9·546		
17	Rheinländer	4·4	14·6	0·08	−1.5	1·0	56	0·84	101	0·06	15·7	16·2	0·33	0·59
18	Ruel	4·61	8·15	0·069	0·74	0·988	35	0·85	69	0·07				
19	Schönauer	4·529	7·772	0·083	0·7324	1·0	34·82	0·8547	68·79	0·0658	9·143	9·641	0·285	0·602
20	Thiele	4·499	8·845	0·0625	0·6389	1·0	41·21	0·8594	65·99	0·0625				
21	Wong	4·50	7·71	0·081	0·762	1·0	34·6	0·858	67·5	0·065	9·15			
22	Betts	4·62	7·93	0·09	0·73	1·00	34·73	0·85(4)	68·31	0·06(7)	9·081*	9·62*	0·284	0·608
23	Donea	4·3124	7·4752	0·09375	0·7296	1	34·670	0·975	70·760	0·0625	9·26	9·65	0·25	0·625
24	Gartling	4·592	7·873	0·0797	0·737	0·9892	34·735	0·8540	68·626	0·0675		9·603	0·2809	0·582
25	Heinrich	4·4309	7·4293	0·0989	0·7337	1·0	35·3722	0·875	70·4538	0·0625				
26	Laval	4·685	8·1814	0·09375	0·7984	1·0	34·596	0·875	69·0347	0·0625	9·145	9·597	0·25	0·625
27	Phuoc	4·08	7·28	0·076	0·47	1·0	34·8	0·86	37·0	0·135				
28	Stevens	4·497	7·86	0·10			34·2	0·855	70·0	0·050	9·10	9·64		
29	Upson I	4·5586	7·8099	0·087	0·728	0·998	34·62	0·856	68·896	0·0663		9·6206	0·272	0·584
30	Upson II	4·5211	7·731	0·0746	0·7277	1·0	34·62	0·856	68·896	0·0663				
31	Winters	4·53	7·74	0·099	0·727	1	34·8	0·86	68·6	0·066	9·117	9·638		
32	Kessler II	4·521	7·744	0·081	0·735	1·00	34·68	0·855	68·40	0·066	9·115	9·625	0·282	0·602
33	Lauriat	4·577	8·117	0	0·718	0·993	35·388	0·854	70·063	0·065				
34	Roux I	4·528	7·738	0·075	0·7362	1	34·70	0·850	67·79	0·075		9·611	0·275	0·600
35	Roux II	4·6548	8·252	0·100	0·7273	1	33·83	0·850	65·96	0·05		9·523	0·400	0·700
36	Roux III													
37	Cooper	4·55	7·69	0·09	0·677	1·0	35·0	0·85	58·5	0·075		10·0		

* 200 elements only.

Table VI. Summary of contributions at $Ra = 10^6$

No.	First author	Nu_0	Nu_{max}	@ $z =$	Nu_{min}	@ $z =$	u_{max}	@ $z =$	w_{max}	@ $x =$	ψ_{mid}	ψ_{max}	@ $x =$	@ $z =$
1	Bertelà	10·56	20·56	0·025	1·095	0·975	64·26	0·875	241·88	0·025	17·60	17·96		
2	Cuvelier													
3	de Vahl Davis	8·797	18·635	0·039	1·065	1·0	64·960	0·850	221·28	0·038	16·41	16·840	0·150	0·547
4	Günther I	9·060	19·44	0·04	0·9006	1	64·65	0·87	217·5	0·04				
5	Günther II	9·136	19·58	0·04	0·9026	1	62·62	0·87	214·70	0·04				
6	Günther III	8·757	18·42	0·04	1·016	1	65·70	0·87	218·70	0·04				
7	Hackbusch													
8	Jones I	8·783	16·89	0·058	1·011	1·0	69·07	0·851	219·5	0·04	17·37	17·81	0·14	0·55
9	Jones II	8·88	18·17	0·035	1·008	1·0	64·17	0·862	220·59	0·0349	16·835	17·256	0·138	0·58
10	Kessler I													
11	Küblbeck	9·467	22·57	0·05	1·54	1	72·53	0·86	220·05	0·04	17·979	18·379	0·16	0·54
12	Le Quere	8·859	18·27	0·04	0·99	1	68·3	0·88	213·0	0·05				
13	Linthorst	8·516	17·00	0·0158	1·674	1	66·58	0·885	222·75	0·036				
14	Portier	9·924	20·761	0·042	0·832	1·00	65·46	0·858	217·20	0·040				
15	Projahn	9·18	19·36	0·039	0·94	1·0	64·0	0·853	219·7	0·033				
16	Quon	8·871	17·765	0·0330	1·011	0·9963	65·69	0·8572	219·74	0·0330	16.262	16·667		
17	Rheinländer	7·8	49	0·08	0	1·0	178	0·86	285	0·046	31·5	39·4	0·23	0·73
18	Ruel	9·31	20·3	0·039	1	0·988	66	0·85	224	0·04				
19	Schönauer	8·844	17·82	0·042	1·048	1·0	65·604	0·850	216·42	0·0373	16·68	17·44	0·145	0·583
20	Thiele	8·695	39·98	0·0625	0·4598	0·94	258·6	0·9219	208·9	0·0468				
21	Wong	8·69	17·36	0·042	1·17	1·00	62·6	0·861	205·0	0·047	16·7			
22	Betts	9·41	16·78	0·07	0·85	0·99	57·67	0·84(5)	213·09	0·05(5)	16·39	16·95	0·167	0·7
23	Donea	8·2341	14·8992	0·0625	0·9632	1	65·698	0·875	226·3	0·03125	16·38	16·54	0·1875	0·5
24	Gartling	9·382	18·630	0·0322	1·007	0·9892	64·368	0·8540	218·424	0·0430		16·851	0·1459	0·582
25	Heinrich	8·344	15·121	0·0625	1·0582	1·0	64·6282	0·875	223·1858	0·03125				
26	Laval	9·6984	19·6426	0·0675	1·5024	1	67·852	0·875	218·226	0·03125	16·83	17·13	0·1875	0·75
27	Phuoc													
28	Stevens	8·767	17·66	0·07			66·7	0·85	221	0·0344	16·51	16·85		
29	Upson I	9·1699	18·508	0·043	0·9837	0·988	64·593	0·850	220·64	0·0316		16·707	0·172	0·5
30	Upson II	8·8170	17·294	0·045	0·9805	1·00	64·593	0·850	220·64	0·0316				
31	Winters	8·83	17·6	0·056	0·975	1	63·9	0·85	222	0·039	16·43	16·80		
32	Kessler II	8·842	17·35	0·045	0·986	1·00	65·21	0·854	220·4	0·039	16·34	16·78	0·151	0·552
33	Lauriat	9·181	21·063	0	1·037	0·995	69·669	0·84236	235·076	0·03499				
34	Roux I													
35	Roux II													
36	Roux III													
37	Cooper	8·88	17·7	0·04	0·36	1·0	65·0	0·87	211	0·04		17·1		

the basis of equal c.p.u. cost, or storage—or even programming effort. However, the first two were not generally reported, and the last is almost impossible to measure.

Having said that, it should be added that we acknowledge that not all contributors have access to unlimited computing time. It is quite noticeable from the results and from the comments of the contributors that government laboratories generally seem to have the better computing facilities; and that many contributors took much time and trouble to participate in what was a peripheral activity for them. Indeed, several were under quite severe pressure to complete other activities (such as doctoral theses) and were unable to devote as much effort to this project as they might have wished.

Quon[10] has gone to some trouble to suggest that a comparison of solutions on the basis of grid point maxima is inadequate and misleading, and that it is necessary to obtain the 'correct functional form' of all field variables. We certainly agree that it would be best to be able to compare *all* features of the contributions; but this is clearly not possible. We sought to compare those aspects of a solution which authors tend to publish (because they have some practical significance). Maxima are important, and are published. If contributors used interpolation to obtain their maxima, so much the better. If not, we could only compare grid point maxima.

Contributors were invited to submit solutions to a specified problem. No restrictions were placed on method, mesh or effort. It was left to contributors to determine these for themselves in ways which would yield the solution. It was left to contributors, for example, to decide how to obtain Nu_{max}: whether or not to use interpolation; if so, which scheme; etc.

It was, of course, clear that a comparison between contributions was to be made. We are thus entitled to consider that contributors have reached their own compromises between accuracy and effort (or cost), and have sent us what they regard as 'the answer to the problem'.

In comparing the contributions, therefore, little consideration has been given to the fact that almost any of the methods would have yielded better solutions if only a finer mesh or a modified method had been used. We have compared what we have been given.

It should be mentioned in this context that the fourth-order deferred correction method of de Vahl Davis and Leong (number 3 in Table II etc.) benefited from inside knowledge: mesh refinement was not used to achieve convergence in isolation, but also to attempt to reproduce the known bench mark solution. It is therefore not entirely surprising that it has performed well. Moreover, the contributions of Jones, Thompson and Woodhouse II (number 9), Linthorst and Schinkel (13), Phuoc and Tanner (27) and Kessler and Oertel (32) were substantially improved in their later submissions.[5b]

The results of Rheinländer differ very considerably from the rest of the results presented in Tables III–VI. The reason for this is that he used a k–ϵ turbulence model, his principal interest being the simulation of time-dependent turbulent flow. He thus, in essence, solved a different problem. Since his results could be of value to others interested in his problem, they have been retained in the tables but have not been used in the comparison with the bench mark solution. It should be noted that his turbulence model was not disabled even at low Ra.

Tables VII–X contain the percentage relative differences between certain characteristics of the contributions and the corresponding characteristics of the bench mark solutions at the different Rayleigh numbers. The positions of the various quantities have been omitted from this comparison, both in order to reduce the volume of data and because they are often difficult to determine accurately, particularly if the maximum is flat. Note that the use of relative error magnifies the errors in Nu_{min} in comparison with the errors in $\overline{\mathrm{Nu}}$ and Nu_{max}. Note also that the bench mark value of $\mathrm{Nu}_{1/2}$ has been used for comparison with the

Table VII. Percentage errors at $Ra = 10^3$

No	First author	Nu	Nu_{max}	Nu_{min}	u_{max}	w_{max}	ψ_{max}
1	Bertelà	0·2	1·0	0·9	0·0	0·4	1·4
2	Cuvelier	−0·5	−1·0	−0·3	−1·3	0·6	
3	de Vahl Davis	−0·2	−0·7	0·7	0·0	0·0	0·0
4	Günther I	0·7	0·4	−1·2	−0·1	−0·5	
5	Günther II	0·0	0·2	−0·2	0·0	−0·1	
6	Günther III	0·2	0·8	−0·8	−0·1	−0·2	
7	Hackbusch	0·0	0·1	0·0	0·0	−0·2	0·9
8	Jones I	−0·1	0·1	0·0	0·0	0·0	0·4
9	Jones II						
10	Kessler I	0·2	1·7	−0·3	1·4	0·1	
11	Küblbeck	0·1	−1·0	5·5	−1·6	−1·3	−0·8
12	Lequere	−0·5	−1·0	1·2	−2·7	−1·5	
13	Linthorst	0·2	0·8	−0·4	−0·5	−0·5	−2·6
14	Portier	2·2	1·3	−1·0	0·3	−0·2	
15	Projahn	2·0	2·3	−2·0	1·4	2·0	
16	Quon	0·9	−0·8	5·2	−1·6	−0·2	−1·5
18	Ruel	0·2	0·3	−0·3	−9·6	0·1	
19	Schönauer	0·0	0·1	0·0	0·0	0·0	−0·1
20	Thiele	0·1	0·4	−0·4	0·0	−0·2	
21	Wong	0·2	−0·3	−0·1	−0·5	−0·5	−0·3
22	Betts	0·2	−0·3	−0·3	−0·2	0·1	0·0
23	Donea	−0·3	−0·1	−0·6	2·2	1·6	0·5
24	Gartling	0·0	0·1	−0·1	−0·2	0·0	0·0
25	Heinrich	−0·8	−1·2	0·8	−4·9	−6·4	
26	Laval	1·8	2·0	−0·2	−7·4	−5·7	0·5
27	Phuoc	1·1	−0·3	−1·7	0·6	0·1	−12·3
28	Stevens	−0·2	−0·5		0·3	0·6	−0·3
29	Upson I	0·0	0·1	−0·1			
30	Upson II	0·0	0·1	−0·1	0·2	0·2	0·1
31	Winters	0·2	0·3	−0·1	−0·2	−0·2	0·1
32	Kessler II	0·0	0·1	−0·1	0·0	0·0	0·1
33	Lauriat	0·0	0·7	−0·7	0·4	0·4	
34	Roux I	0·0	0·3	−0·3	−0·1	−0·2	−0·1
35	Roux II	−0·2	−0·6	0·7	−0·6	−1·2	−0·3
36	Roux III	0·6	2·7	−1·3	−0·7	−1·3	−0·4
37	Cooper	0·2	1·0	−1·7	2·8	−1·8	−0·3

contributed average Nusselt numbers, since it is believed to be the most accurate estimate of the average.

The first general conclusion to be drawn is that although there are accurate contributions using both FEM and FDM, the former are by and large rather better. There is a lower tendency among the FEM entries towards a degradation of performance with increasing Rayleigh number and a much lower number of contributions containing obvious major errors. The FEM was also better able to cope with the higher Rayleigh number, only one (27) failing to supply answers for $Ra = 10^6$ (the stated reason being cost).

It is tempting to attribute this superior performance to the more prevalent use of a non-uniform distribution of grid points in the FEM. However, those FDM which have used a non-uniform mesh (9, 12–16, 18 and 19) have not, on the whole, performed better than those which used only a uniform mesh. This conclusion was very surprising to us, since there

Table VIII. Percentage errors at Ra = 10^4

No.	First author	Nu	Nu_{max}	Nu_{min}	u_{max}	w_{max}	ψ_{max}
1	Bertelà	2·5	5·4	−1·0	−0·2	−0·2	1·0
2	Cuvelier	1·1	0·1	−7·8	−3·3	−0·7	
3	de Vahl Davis	0·0	−0·6	0·5	0·1	0·1	0·2
4	Günther I	0·2	−0·1	−1·9	−0·2	−0·3	
5	Günther II	0·3	0·8	−0·7	−0·4	−0·5	
6	Günther III	0·6	0·9	0·9	−0·5	−0·5	
7	Hackbusch	0·4	1·1	−0·4	6·1	−0·4	1·1
8	Jones I	0·0	0·6	0·0	0·3	0·1	0·8
9	Jones II						
10	Kessler I	1·2	4·0	7·5	−0·5	−0·6	
11	Küblbeck	2·2	3·5	10·9	0·4	−1·5	1·4
12	Lequere	−0·5	0·6	4·1	−3·9	−2·3	
13	Linthorst	0·6	1·2	0·5	−0·9	−0·6	−2·4
14	Portier	5·6	3·7	−1·4	0·1	−0·4	
15	Projahn	1·2	3·2	0·7	0·1	−0·3	
16	Quon	1·0	−2·6	16·0	−0·5	−0·3	−0·4
18	Ruel	0·8	25·0	22·9	−1·1	2·0	
19	Schönauer	0·2	0·4	−0·3	0·1	−0·4	0·0
20	Thiele	0·8	1·9	−1·9	0·7	−1·0	
21	Wong	−0·1	0·1	0·5	−0·5	−0·6	−0·2
22	Betts	2·1	2·0	0·7	−0·5	1·5	0·0
23	Donea	1·1	0·3	−0·6	3·2	2·1	2·5
24	Gartling	0·3	0·3	0·2	0·0	0·1	0·1
25	Heinrich	−2·1	−3·0	−0·4	−2·0	1·6	
26	Laval	10·3	6·2	9·8	−5·8	1·8	0·7
27	Phuoc	−1·0	−0·5	−21·5	0·3	−0·5	2·0
28	Stevens	−0·1	0·1		0·1	0·4	0·4
29	Upson I	0·2	0·2	−0·1			
30	Upson II	0·1	0·2	−0·2	0·1	0·3	0·0
31	Winters	0·3	0·3	−0·2	0·1	0·4	0·1
32	Kessler II	0·1	0·0	0·0	0·1	0·1	0·1
33	Lauriat	0·2	1·5	−1·4	0·5	1·5	
34	Roux I	0·3	0·5	−0·6	−0·1	0·0	0·0
35	Roux II	0·1	0·5	0·5	−2·0	−1·9	−0·7
36	Roux III						
37	Cooper	0·8	2·9	−3·9	−1·1	−2·6	0·4

is a popular theory (to which we subscribe) that a denser distribution of mesh points in suitably chosen locations will lead to improved accuracy—provided, of course, that the consequent coarse distribution elsewhere does not introduce a countervailing contamination of the solution.

Most of the non-uniform grid results, however, have been obtained using fewer grid points than were used with the finest uniform grids. This presumably gives roughly the same number of grid points within the boundary layers for the higher Rayleigh number cases. This matter has also been discussed in detail by Quon[10] in a comparison of his own results with some of our contributions. It is clear from the present study and from Quon's paper that there is scope for more work on the use of co-ordinate stretching and selective mesh refinement at high Rayleigh numbers for both finite difference and finite element methods.

Some of the FEM tended to do well on velocities, but not so well on heat transfer rates,

Table IX. Percentage errors at $Ra = 10^5$

No.	First author	Nu	Nu_{max}	Nu_{min}	u_{max}	w_{max}	ψ_c	ψ_{max}
1	Bertelà	8·4	16·9	−5·3	−0·2	−2·9	3·2	3·0
2	Cuvelier	6·2	12·6	−46·5	−3·5	−4·1		
3	de Vahl Davis	0·0	−1·5	1·4	0·2	0·1	0·1	0·1
4	Günther I	1·2	0·0	−6·8	13·6	2·2		
5	Günther II	1·0	2·9	−2·4	−0·1	−1·2		
6	Günther III	0·4	1·7	1·9	1·0	−1·0		
7	Hackbusch	1·8	6·2	−2·1	1·3	−0·9	3·2	4·0
8	Jones I	0·1	1·3	0·7	1·8	0·1	1·9	2·0
9	Jones II							
10	Kessler I							
11	Küblbeck	2·6	8·5	27·6	5·8	−1·0	6·4	5·1
12	Lequere	−1·1	1·3	12·5	4·7	−0·2		
13	Linthorst	0·4	−0·1	8·4	0·3	0·4	4·5	−4·8
14	Portier	7·7	8·7	−3·3	1·2	−0·7		
15	Projahn	3·1	4·8	0·1	0·2	0·3		
16	Quon	0·4	−1·9	10·8	0·2	−0·5	−0·1	−0·7
18	Ruel	2·0	5·6	1·5	0·8	0·6		
19	Schönauer	0·2	0·7	0·5	0·3	0·3	0·4	0·3
20	Thiele	−0·4	14·6	−12·4	18·7	−3·8		
21	Wong	−0·4	−0·1	4·5	−0·4	−1·6	0·4	
22	Betts	2·2	2·8	0·1	0·0	−0·4	−0·3	0·1
23	Donea	−4·6	−3·1	0·1	−0·2	3·2	1·6	0·4
24	Gartling	1·6	2·0	1·1	0·0	0·1		−0·1
25	Heinrich	−1·9	−3·7	0·6	1·8	2·7		
26	Laval	3·7	6·0	9·5	−0·4	0·6	0·4	−0·2
27	Phuoc	−9·7	−5·7	−35·5	2·1	5·2	−3·1	−1·2
28	Stevens	−0·5	1·9		−1·5	2·1	−0·1	0·3
29	Upson I	0·9	1·2	−0·1	−0·3	0·4		0·1
30	Upson II	0·0	0·2	−0·2	−0·3	0·4		0·1
31	Winters	0·2	0·3	−0·3	0·2	0·0	0·1	0·3
32	Kessler II	0·0	0·3	0·8	−0·1	−0·3	0·0	0·1
33	Lauriat	1·3	5·2	−1·5	1·9	2·1		
34	Roux I	0·2	0·3	1·0	−0·1	−1·2		0·0
35	Roux II	3·0	6·9	−0·2	−2·6	−3·8		−0·9
36	Roux III							
37	Cooper	0·7	−0·3	−7·1	0·8	−14·7		4·0

particularly at the higher Rayleigh numbers, e.g. Betts and Lidder (22), Gartling (24), Donea and Giuliani (23) and Laval (26). The consistent flux method of Upson *et al.* (30) gave more accurate mean Nusselt numbers than their more conventional Gauss point method (29). This was also true for their values of Nu_{max}, except at $Ra = 10^6$. As already noted, we are less confident of the accuracy of the bench mark value of Nu_{max} at 10^6 than we are of the other characteristics of the bench mark solution. And we note that two contributions—those of Upson *et al.* (30) and Kessler and Oertel (32)—which otherwise agree well with the bench mark, have values which agree with each other (within 0·3 per cent), suggesting a value for Nu_{max} of about 17·3, some 3·5 per cent below the bench mark value. However, two other solutions—those of Quon (16) and Winters (31)—which are almost as good, support a larger value (say 17·7, or 1·3 per cent below the bench mark value). Stevens (28) and Winters (31) also consistently obtained accurate mean Nusselt numbers with their methods.

Table X. Percentage errors at $Ra = 10^6$

No.	First author	Nu	Nu_{max}	Nu_{min}	u_{max}	w_{max}	ψ_c	ψ_{max}
1	Bertelà	20·0	14·7	10·7	−0·6	10·3	7·8	7·2
2	Cuvelier							
3	de Vahl Davis	0·2	4·0	7·7	0·5	0·9	0·6	0·5
4	Günther I	3·0	8·5	−8·9	0·0	−0·8		
5	Günther II	3·8	9·2	−8·7	−3·1	−2·1		
6	Günther III	−0·5	2·8	2·7	1·7	−0·3		
7	Hackbusch							
8	Jones I	−0·2	−5·8	2·2	6·9	0·1	6·4	6·3
9	Jones II	0·9	1·4	1·9	−0·7	0·6	3·2	3·0
10	Kessler I							
11	Küblbeck	7·6	25·9	55·7	12·2	0·3	10·2	9·7
12	Lequere	0·6	1·9	0·1	5·7	−2·9		
13	Linthorst	−3·2	−5·2	69·3	3·0	1·5	1·7	−2·4
14	Portier	12·8	15·8	−15·9	1·3	−1·0		
15	Projahn	4·3	8·0	−5·0	−1·0	0·2		
16	Quon	0·8	−0·9	2·2	1·6	0·2	−0·4	−0·5
18	Ruel	5·8	13·2	1·1	2·1	2·1		
19	Schönauer	0·5	−0·6	6·0	1·5	−1·3	2·2	4·1
20	Thiele	−1·2	123·0	−53·5	300·1	−4·8		
21	Wong	−1·2	−3·2	18·3	−3·1	−6·5	2·3	
22	Betts	6·9	−6·4	−14·1	−10·8	−2·9	0·4	1·2
23	Donea	−6·4	−16·9	−2·6	1·7	3·2	0·4	−1·3
24	Gartling	6·6	3·9	1·8	−0·4	−0·4		0·6
25	Heinrich	−5·2	−15·6	7·0	0·00	1·7		
26	Laval	10·2	9·6	51·9	5·0	−0·5	3·1	2·3
27	Phuoc							
28	Stevens	−0·4	−1·5		3·2	0·7	1·2	0·6
29	Upson I	4·2	3·3	−0·5	−0·1	0·6		−0·3
30	Upson II	0·2	−3·5	−0·9	−0·1	0·6		−0·3
31	Winters	0·4	−1·8	−1·4	−1·1	1·2	0·7	0·3
32	Kessler II	0·5	−3·2	−0·3	0·9	0·5	0·1	0·2
33	Lauriat	4·3	17·5	4·9	7·8	7·2		
34	Roux I							
35	Roux II							
36	Roux III							
37	Cooper	0·9	−1·3	−63·6	0·6	−3·8		2·1

There is too much scatter in the finite difference results to enable any general conclusions to be drawn. For example, Portier *et al.* (14) and de Vahl Davis and Leong (3) gave results which have larger errors for Nu than for the velocities whereas the results for Jones *et al.* (8, 9) and Le Quere and Humphrey (12) generally had more accurate values for Nu.

Several contributions are, at least for some values of Ra, in general agreement with the bench mark but contain one or two features which are significantly in error. These include Günther's (4) value of u_{max} at $Ra = 10^5$; Ruel, Grand and Latrobe's (18) value of u_{max} at 10^3, and their Nu_{max} and Nu_{min} at 10^4; Phuoc and Tanner's (27) value of ψ_{mid} at 10^3; and Quon's (16) value of Nu_{min}—a difficult quantity to obtain accurately—at each Ra except, curiously, at 10^6, where his value is quite good.

Many contributions suffer from declining quality with increasing Rayleigh number, often despite the accompanying and counteracting use of mesh refinement. Particularly notable in this respect are the contributions of Bertelà (1), Küblbeck and Straub (11), Thiele (20) and Lauriat (33), although several others display this defect to a lesser extent. Thiele had this difficulty despite going to a 65×65 mesh; he was also unable to obtain a steady solution at $\mathrm{Ra} = 10^6$.

The non-Boussinesq ideal gas formulation of Le Quere and Humphrey (12) gave answers which were generally in good agreement with those for a Boussinesq fluid, particularly for the overall Nusselt number. They used 293 K and 273 K as the hot and cold boundary temperatures respectively; this temperature difference is below but close to the limit of validity[11] of the Boussinesq approximation used in the bench mark and all other solutions.

Roux, Bontoux, Gilly and Grondin used three methods (34–36). The first, fully fourth-order, is the most accurate but was only employed at $\mathrm{Ra} \leqslant 10^5$; it would be interesting to see if its accuracy is maintained at $\mathrm{Ra} = 10^6$. With a return to a second-order method for ζ and then also (at $\mathrm{Ra} = 10^3$ only) for T, the quality of their results deteriorated.

The results of Hackbusch (7) are consistently fair, apart from $u_{\max}$ at 10^4 and $\mathrm{Nu}_{\max}$ at 10^5, which are high. His method—using the biharmonic equation for ψ—has the attraction of avoiding the need for a vorticity boundary condition. However, it was unable to yield a solution at $\mathrm{Ra} = 10^6$. It was also the only method that was noticeably very fast, e.g. 3.8 s on a CDC Cyber 70/76 for a 33×33 grid at $\mathrm{Ra} = 10^5$, including plotting time. Solution times could have been reduced even further with the multi-grid method by requiring only two digits of accuracy.

The method of Laval (26) achieved an accuracy which was more or less independent of Rayleigh number. The accuracy is not high—it is of the order of several per cent—but it is reasonable in the light of the coarse mesh used. Projahn and Rieger (15) also obtained results of fairly uniform (and somewhat better) quality. Günther's third method (6) did even better, although there appears to be a slight deterioration of quality with increasing Ra. The fourth-order deferred correction method of de Vahl Davis and Leong (3) did well, with the benefit of knowledge of the bench mark solution as a goal.

Quon (16), as mentioned above, had difficulty with $\mathrm{Nu}_{\min}$, but was otherwise within one or two per cent of the accurate solution, as were Jones *et al.* (8, 9), Stevens (28), Winters (31) and Cooper and Pepper (37). Quon[10] has also published additional results using a 60×60 non-uniform grid for $\mathrm{Ra} = 10^6$. They are better than his values which are presented here and their accuracy is high. Gartling's velocities (24) were accurate but his Nusselt numbers were not good at 10^6.

Schönauer and Raith (19), and Wong and Raithby (21) each estimated the accuracy of their submissions to be about 1 per cent at $\mathrm{Ra} = 10^5$; Schönauer and Raith also predicted a 4 per cent error at 10^6. If these two solutions are compared with the bench mark, the error estimates will be seen to be quite reliable.

The best contributed results were achieved by the FEM of Upson, Gresho and Lee (29, 30) and the Galerkin method of Kessler and Oertel (32). In most cases their results agree with ours to better than 1 per cent. The average and extreme values of wall heat flux at the higher Rayleigh numbers were the characteristics which agreed least well. It is perhaps significant that Upson *et al.* took considerable care to provide a high density of grid points in the wall and corner regions of the cavity.

CONCLUSIONS

We have presented here a summary of the results of a comparison exercise which is intended to provide a basis for the assessment of numerical methods for the solution of problems of buoyancy-driven flow. It is extremely gratifying that so many of the numerical results are substantially in agreement with each other, and that those which one would expect to give more accurate results (e.g. higher order methods and those with more grid points) do so. This, in itself, has enabled the principal aim of our exercise to be met: the generation of a high accuracy solution which has been validated by several independent calculations. We feel that this agreement between solutions allows us to assert with some confidence that the bench mark solution, and of course those which agree with it closely, represent an accurate solution of the problem. It is to be hoped that those wishing in the future to verify their algorithms and programs will compute results for the same standard set of parameters to enable an objective assessment to be obtained. It is also to be hoped that our contributors, and others interested in problems of this type, will now set out to improve their existing algorithms and throw light on many of the interesting features that have emerged from this study. Certainly very few of us have any cause to be complacent.

It is invidious to seek best methods or winners from an exercise such as this; and it was not our aim to do so. It is, however, pleasurable to report that the three most accurate results for all the parameter values were provided by a finite element method, a Galerkin method and a finite difference method, viz. the contributions of Upson, Gresho and Lee (29, 30) and of Kessler and Oertel (32) together with the bench mark itself.[9] Many others were in close pursuit but, because of the scatter in the errors, it is not possible to separate them into different categories.

Finally, we would again like to express our sincere thanks to the contributors and to many others for their active encouragement, interest and patience throughout the course of this exercise.

APPENDIX

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NOTE ADDED IN PROOF:

Since the submission of this paper, two additional computations have come to our attention.

(1) P. Le Quere and T. Alziary de Roquefort (*C.R. Acad. Sc. Paris*, **294,** Series II, 941–944, 1982) solved the PV equations using Chebychev polynomials with a semi-implicit spectral method. With 17×17 nodes, their results were good: errors of less than 0·3 per cent in the variables of Tables VII–X at $Ra \leqslant 10^5$, although up to several per cent at 10^6; with 33×33 nodes they were excellent: less than 0·2 per cent for $Ra \leqslant 10^5$ and (except for Nu_{max}), less than 1 per cent at 10^6. Their value of Nu_{max} was 17·553 (with 33×33 nodes), somewhere between the bench mark value and the values computed by Upson *et al.* (30) and Kessler *et al.* (32).

(2) N. C. Markatos and K. A. Pericleous (*Report PDR/CHAM UK/16*, Cham Ltd, London, 1982) also solved the PV equations using an upwind finite domain code implemented in a commercially available program called PHOENICS. The grids used were 30×30 at 10^3; 40×40 at 10^4 and 10^5; and 80×80 at 10^6; all non-uniform. The results are only fair with errors in velocities of 5 per cent at $Ra = 10^3$, up to 12 per cent in u_{max} at 10^6 (but only 0·2 per cent in w_{max} at 10^6). Errors in $Nu_{1/2}$ and Nu_{max} range from 1 to 7 per cent; they appear to have been obtained by graphical, rather than numerical, differentiation. Errors in Nu_{min}, not surprisingly under those circumstances, are up to 23 per cent.